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Global structure of plane closed elastic curves

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1 Introduction

This is a joint work with Waichiro Matsumoto and Shoji Yotsutani (Ryukoku University).

Let Γ be a plane closed elastic curve with length 2π. We denote arc-length and curvature by s and $\kappa(s)$, respectively. Let M be the signed area defined by

$$M:=\frac{1}{2}\int_{\Gamma}xdy-ydx,$$

where $(x,y)=(x(s),y(s))\in\Gamma$ with $(x(0),y(0)):=(0,0)$. Let us consider the following variational problem (VP):

Find a curve Γ *(the curvature* $\kappa(s)$*) which minimize* $\frac{1}{2}\int_{0}^{2\pi}\kappa(s)^{2}ds$ *subject to* $\pi>M$ *and* $\omega\pi\neq M$*, where* ω *is the winding number.*

K. Watanabe ([1, 2]) considered this variational problem (VP) with $\omega=1$. He derived the Euler-Lagrange equation to (VP) and showed the existence of the minimizer and investigate the profile near the disk.

The Euler-Lagrange equation to (VP) is

$$(P^{\omega})\begin{cases}\kappa_{ss}+\frac{1}{2}\kappa^{3}+\mu\kappa-\nu=0, \quad s\in[0,2\pi],\\ \kappa(0)=\kappa(2\pi),\ \kappa_{s}(0)=\kappa_{s}(2\pi),\\ \frac{1}{2\pi}\int_{0}^{2\pi}\kappa(s)ds=\omega,\\ \dfrac{4\mu\pi^{2}+\pi\int_{0}^{2\pi}\kappa(s)^{2}ds}{4\pi\omega\mu+\int_{0}^{2\pi}\kappa(s)^{3}ds}=M,\end{cases}$$

where μ and ν are some constants. We can obtain the following proposition by using the argument of K.Watanabe [1, Lemma 3 and Lemma 4]

Proposition 1.1 *Suppose that* $\kappa(s)$ *is a solution of* (P^{ω})*, then the following properties hold:*
(i) $\kappa(s)\in C^{\infty}([0,2\pi])$.
(ii) There exists a positive integer m *such that* $\kappa(s)$ *is periodic function*

with period $s = 2\pi/m$ and axially symmetric with respect to $s = \pi/m$ and m denotes the number of minimum points of $\kappa(s)$ by normalizing $\kappa(0) := \max_{0\le s\le 2\pi}\kappa(s)$. (We call this solution "m − mode solution".)

Let us normalize $\kappa(s)$ as $\kappa(0) := \max_{0\le s\le 2\pi}\kappa(s)$. For n-mode solution $\kappa(s)$, we may consider the following differential equation:

$$
(P_n^\omega)\begin{cases}
\kappa_{ss} + \dfrac{1}{2}\kappa^3 + \mu\kappa - \nu = 0, \quad s \in \left[0, \dfrac{\pi}{n}\right], & (1.1)\\
\kappa_s(0) = \kappa_s\left(\dfrac{\pi}{n}\right) = 0,\ \kappa_s(s) < 0 \ \ s \in \left(0, \dfrac{\pi}{n}\right), & (1.2)\\
\displaystyle\int_0^{\pi/n} \kappa(s)ds = \frac{\omega\pi}{n}, & (1.3)\\
\dfrac{2\mu\pi^2 + n\pi \displaystyle\int_0^{\pi/n)} \kappa(s)^2 ds}{2\pi\omega\mu + n\displaystyle\int_0^{\pi/n} \kappa(s)^3 ds} = M. & (1.4)
\end{cases}
$$

We introduce the following auxiliary problem. Let $\kappa(s)$ be unknown function, and μ, ν be unknown constants. Find $(\kappa(s), \mu, \nu)$ such that

$$
(E_n)\begin{cases}
\kappa_{ss} + \dfrac{1}{2}\kappa^3 + \mu\kappa - \nu = 0, \quad s \in \left[0, \dfrac{\pi}{n}\right], & (1.5)\\
\kappa_s(0) = \kappa_s\left(\dfrac{\pi}{n}\right) = 0,\ \kappa_s(s) < 0 \ for\ s \in \left(0, \dfrac{\pi}{n}\right). & (1.6)
\end{cases}
$$

First we represent all solution $(\kappa(s), \mu, \nu)$ of (E_n). Next we give the representation of the constraint (1.3) and (1.4).

We prepare notations to state our theorems.

Definition 1.1 *We define the complete elliptic integral of first, second and third kind by*

$$
\begin{aligned}
K(k) &:= \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}},\\
E(k) &:= \int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}d\xi,\\
\Pi(\ell, k) &:= \int_0^1 \frac{d\xi}{(1+\ell\xi^2)\sqrt{(1-\xi^2)(1-k^2\xi^2)}}.
\end{aligned}
$$

Definition 1.2 *Jacobi's* sn *function is defined by*

$$z = \int_0^{\mathrm{sn}(z,k)} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}$$

and Jacobi's cn *function is defined by*

$$\mathrm{cn}(z,k) := \sqrt{1-\mathrm{sn}^2(z,k)}$$

for $z \in [-K(k), K(k)]$. *These elliptic functions are extended to* $(-\infty,\infty)$ *by using the relation* $\mathrm{sn}(z+2K(k),k) = -\mathrm{sn}(z,k)$ *and* $\mathrm{cn}(z+2K(k),k) = -\mathrm{cn}(z,k)$.

1.1 Main Result

Theorem 1.1 *All solutions* $(\kappa(s),\mu,\nu)$ *of* (E_n) *are represented by the following (i), (ii) and (iii):*

(i) $\kappa(s) = \overline{\kappa}_n(s;k,h), \mu = \overline{\mu}_n(k,h)$ *and* $\nu = \overline{\nu}_n(k,h)$ *for* $(k,h) \in \overline{\Sigma}$, *where*

$$\overline{\Sigma} := \Sigma_{S^*} \cup \Sigma_S, \tag{1.7}$$

$$\Sigma_{S^*} := \{(k,h);\ -1 < k \le 0,\ 2 < h < 3\}, \tag{1.8}$$

$$\Sigma_S := \{(k,h);\ 0 \le k < 1,\ 0 < h \le 3-2k^2\}, \tag{1.9}$$

$$\overline{\kappa}_n(s;k,h) := \begin{cases} \kappa_n^{S^*}(s;k,v(k,h)) & \text{for } (k,h) \in \Sigma_{S^*}, \\ \kappa_n^{S}(s;k,u(k,h)) & \text{for } (k,h) \in \Sigma_S, \end{cases} \tag{1.10}$$

$$\overline{\mu}_n(k,h) := \begin{cases} \mu_n^{S^*}(k,v(k,h)) & \text{for } (k,h) \in \Sigma_{S^*}, \\ \mu_n^{S}(k,u(k,h)) & \text{for } (k,h) \in \Sigma_S, \end{cases} \tag{1.11}$$

and

$$\overline{\nu}_n(k,h) := \begin{cases} \nu_n^{S^*}(k,v(k,h)) & \text{for } (k,h) \in \Sigma_{S^*}, \\ \nu_n^{S}(k,u(k,h)) & \text{for } (k,h) \in \Sigma_S. \end{cases} \tag{1.12}$$

Here the functions $\kappa_n^*(s;k,v), \mu_n^*(k,v), \nu_n^*(k,v)$ *and* $v(k,h)$ *are defined by*

$$\kappa_n^{S^*}(s;k,v) := -\frac{\sqrt{1-v}\sqrt{(1-k^2)v+1+k^2}}{\sqrt{v+1}\left(2-(1+v)\mathrm{cn}^2\left(\frac{n}{\pi}K(k)\left(\frac{\pi}{n}-s\right),k\right)\right)}\left(\frac{4\sqrt{2}n}{\pi}K(k)\right) \tag{1.13}$$

$$+\frac{4-(1-k^2)(1-v)^2+4k^2(1-v)}{\sqrt{1-v^2}\sqrt{(1-k^2)v+1+k^2}}\left(\frac{n}{\sqrt{2}\pi}K(k)\right),$$

$$\mu_n^{S^*}(k,v) := \left(\frac{-3\big(4-(1-k^2)(1-v)^2\big)^2}{(1-v^2)\big((1-k^2)v+1+k^2\big)} + 8(2-k^2)\right)\left(\frac{n}{2\pi}K(k)\right)^2, \tag{1.14}$$

$$\nu_n^{S^*}(k,v) := \frac{-2\sqrt{2}\big(4-(1-k^2)(1-v)^2\big)}{(1-v^2)^{3/2}\big((1-k^2)v+1+k^2\big)^{3/2}} \cdot \left((1+v)^2\big((1-k^2)v+1+k^2\big)^2 - k^4(1-v)^2\right)\left(\frac{n}{2\pi}K(k)\right)^3 \tag{1.15}$$

and

$$v(k,h) := \frac{-2+(2-k^2)(2-h)+\sqrt{(2-k^2)^2(2-h)^2+4k^4(3-h)}}{2(1-k^2)} \tag{1.16}$$

and the functions $\kappa_n(s;k,u), \mu_n(k,u), \nu_n(k,u)$ *and* $u(k,h)$ *are also defined by*

$$\kappa_n^S(s;k,u) := -\frac{(1-k^2)(1-ku)+k\big((1-k^2)u+k\big)\operatorname{cn}\left(\frac{2n}{\pi}K(k)\left(\frac{\pi}{n}-s\right),k\right)}{(1-k^2)u+k-k(1-ku)\operatorname{cn}\left(\frac{2n}{\pi}K(k)\left(\frac{\pi}{n}-s\right)s,k\right)} \cdot \frac{\sqrt{u}\sqrt{(1-2k^2)u+2k}}{\sqrt{(1-k^2)u^2+1}} \cdot \left(\frac{4nK(k)}{\pi}\right) + \frac{(1-ku)\big((1-k^2)u+k\big)}{\sqrt{u}\sqrt{(1-2k^2)u+2k}\sqrt{(1-k^2)u^2+1}}\left(\frac{2nK(k)}{\pi}\right), \tag{1.17}$$

$$\mu_n^S(k,u) := \left(\frac{-3(1-ku)^2\big((1-k^2)u+k\big)^2}{2u\big((1-2k^2)u+2k\big)\big((1-k^2)u^2+1\big)} + 1 - 2k^2\right) \cdot \left(\frac{2nK(k)}{\pi}\right)^2, \tag{1.18}$$

$$\nu_n^S(k,u) := \frac{-(1-ku)\big((1-k^2)u+k\big)}{4u^{3/2}\big((1-2k^2)u+2k\big)^{3/2}\big((1-k^2)u^2+1\big)^{3/2}} \cdot \left(4k^2\big((1-k^2)u^2+1\big)^2 + (1-k^2)u^2\big((1-2k^2)u+2k\big)^2\right) \cdot \left(\frac{2nK(k)}{\pi}\right)^3 \tag{1.19}$$

and

$$u(k,h) := \frac{1}{4k(1-k^2)} \cdot \left(2-h+ \frac{(1-2k^2)\big((2-h)^2+16k^2(1-k^2)\big)}{8k^2(1-k^2)+\sqrt{(1-2k^2)^2(2-h)^2+16k^2(1-k^2)}}\right). \tag{1.20}$$

(ii) $\kappa(s) = \underline{\kappa}_n(s;k,h), \mu = \underline{\mu}_n(k,h)$ *and* $\nu = \underline{\nu}_n(k,h)$ *for* $(k,h) \in \underline{\Sigma}$, *where*

$$\begin{aligned} \underline{\Sigma} \ &:= \Sigma_{R^*} \cup \Sigma_R, && (1.21)\\ \Sigma_{R^*} &:= \{(k,h);\ -1 < k \le 0,\ -3 < h < -2\}, && (1.22)\\ \Sigma_R \ &:= \{(k,h);\ 0 \le k < 1,\ 2k^2 - 3 \le h < 0\}. && (1.23) \end{aligned}$$

Here

$$\begin{aligned} \underline{\kappa}_n(s;k,h) &:= -\overline{\kappa}_n\left(\frac{\pi}{n} - s, k, -h\right), \\ \underline{\mu}_n(k,h) &:= \overline{\mu}_n(k,-h), \quad \underline{\nu}_n(k,h) := -\overline{\nu}_n(k,-h). \end{aligned} \qquad (1.24)$$

(iii) $\kappa(s) = \overline{\kappa}_n(s;k,h), \mu = \overline{\mu}_n(k,h)$ *and* $\nu = \overline{\nu}_n(k,h)$ *for* $(k,h) \in \Sigma_0$, *where*

$$\Sigma_0 := \{(k,h);\ 0 < k < 1, h = 0\}. \qquad (1.25)$$

Here

$$\overline{\kappa}_n(s;k,h) = \frac{4nkK(k)}{\pi}\mathrm{cn}\left(\frac{2n}{\pi}K(k)s, k\right),$$

$$\overline{\mu}_n(k,h) = (1-2k^2)\left(\frac{2nK(k)}{\pi}\right)^2, \ \overline{\nu}_n(k,h) = 0.$$

We show the domains $\overline{\Sigma} \cup \underline{\Sigma} \cup \Sigma_0$ in Figure 1.

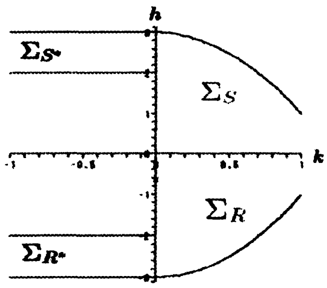

Figure 1: The domain of $\overline{\Sigma} \cup \underline{\Sigma} \cup \Sigma_0$

Remark 1.1 *It is more useful by using the parameter* (k,u) *and* (k,v) *than* (k,h). *Let us set*

$$\begin{aligned} \Sigma_v &:= \{(k,v);\ -1 < k \le 0, -1 < v < 1/k\}, && (1.26)\\ \Sigma_u &:= \{(k,u);\ 0 < k < 1, 0 < u < 1/k\}. && (1.27) \end{aligned}$$

Then the following (i),(ii),(iii) and (iv) hold:
(i) Changing the parameters from (k,h) *to* (k,v) *by* $k = k$ *and* $v = v(k,h)$, Σ_{S^*} *becomes* Σ_v.

(ii) Changing the parameter from (k,h) *to* (k,u) *by* $k=k$ *and* $u=u(k,h)$, Σ_S *becomes* Σ_u.

(iii) Changing the parameter from (k,h) *to* (k,u) *by* $k=k$ *and* $u=u(k,-h)$, Σ_R *becomes* Σ_u.

(iv) Changing the parameter from (k,h) *to* (k,v) *by* $k=k$ *and* $v=v(k,-h)$, Σ_{R^*} *becomes* Σ_v.

We note that all changing the parameters are bijective.

We show the domains Σ_v and Σ_u in Figure 2.

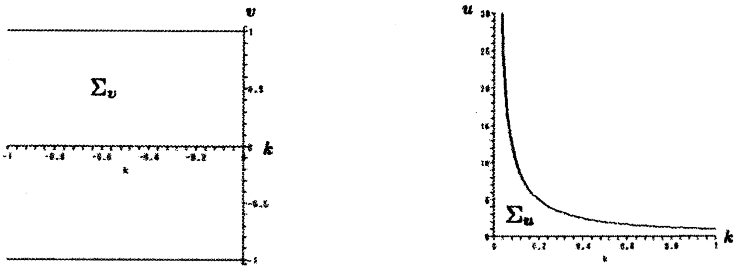

Figure 2: The domains Σ_v and Σ_u

Theorem 1.2 *Let* $\kappa(s)$ *be given by Theorem 1.1 and*

$$Z(k,h) := \int_0^{\pi/n} \kappa(s)ds.$$

Then $Z(k,h)$ *is given by the following (i), (ii) and (iii):*
(i) $Z(k,h)=\overline{Z}(k,h)$ *for* $(k,h)\in\overline{\Sigma}$, *where*

$$\overline{Z}(k,h) := \begin{cases} Z^{S^*}(k,v(k,h)) \ \textit{for} \ (k,h)\in\Sigma_{S^*}, \\ Z^S(k,u(k,h)) \ \textit{for} \ (k,h)\in\Sigma_S. \end{cases}$$

Here the function $Z^{S^*}(k,v)$ *is defined by*

$$Z^{S^*}(k,v) := \frac{Z^*_\infty(k,v)}{Z^*_0(k,v)}, \tag{1.28}$$

$$\begin{aligned} Z^*_\infty(k,v) :=& \big((1-k^2)(1+v)(3-v)+4k^2v\big)K(k) \\ &-4(1-k^2)(1-v^2)\Pi\left(\frac{1}{2}(1-k^2)(1-v)-1,k\right), \end{aligned} \tag{1.29}$$

$$Z^*_0(k,v) := \sqrt{2}\sqrt{1-v^2}\sqrt{(1-k^2)v+1+k^2} \tag{1.30}$$

and the function $Z^S(k,u)$ is also defined by

$$Z^S(k,u) := \frac{2\big((1-k^2)u+k\big)Z_\infty(k,u)}{Z_0(k,u)}, \tag{1.31}$$

$$Z_\infty(k,u) := \big(2(1-k^2)u^2+2-(1-ku)^2\big)K(k) \tag{1.32}$$
$$\qquad - 2\big((1-k^2)u^2+1\big)\Pi\left(\frac{k^2(1-ku)^2}{u\big((1-2k^2)u+2k\big)}, k\right),$$

$$Z_0(k,u) := (1-ku)\sqrt{u}\sqrt{(1-2k^2)u+2k}\sqrt{(1-k^2)u^2+1}. \tag{1.33}$$
$$\tag{1.34}$$

(ii) $Z(k,h) = \underline{Z}(k,h)$ for $(k,h) \in \underline{\Sigma}$, where

$$\underline{Z}(k,h) := -\overline{Z}(k,-h).$$

(iii) $Z(k,h) = 0$ for $(k,h) \in \Sigma_0$.

The relation $Z(k,h)$ is represented by two forms $\overline{Z}(k,h)$ and $\underline{Z}(k,h)$. The relation $Z(k,h) = \dfrac{\omega\pi}{n}$ is equivalent to (1.3). For example, we show the level curves of $Z(k,h) = 0$ in the case $\omega = 0$ in Figure 3.

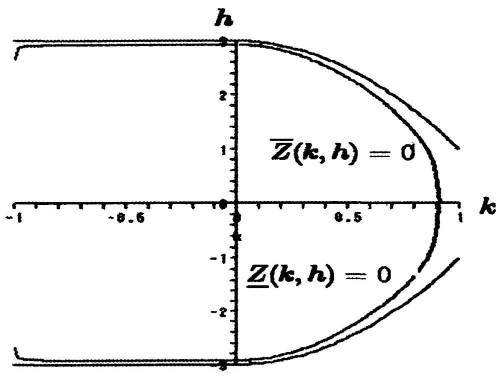

Figure 3: The level curves of $Z(k,h) = 0$

Theorem 1.3 *Let $\kappa(s)$ be given by Theorem 1.1 and*

$$\mathcal{E}_n(k,h) := n\int_0^{\pi/n} \kappa(s)^2 ds.$$

Then $\mathcal{E}_n(k,h)$ is given by the following (i),(ii) and (iii):
(i) $\mathcal{E}_n(k,h) = \overline{\mathcal{E}}_n(k,h)$ for $(k,h) \in \overline{\Sigma}$, where

$$\overline{\mathcal{E}}_n(k,h) := \begin{cases} \mathcal{E}_n^{S^*}(k, v(k,h)) & for\ (k,h) \in \Sigma_{S^*}, \\ \mathcal{E}_n^{S}(k, u(k,h)) & for\ (k,h) \in \Sigma_{S}. \end{cases} \tag{1.35}$$

Here the function $\mathcal{E}_n^{S^*}(k,v)$ is defined by

$$\mathcal{E}_n^{S^*}(k,v) := \frac{2n^2}{\pi} K(k) \cdot \left(\left(\frac{\left(4-(1-k^2)(1-v)^2\right)^2}{(1-v^2)\left((1-k^2)v+k^2+1\right)} + 8k^2 - 16 \right) K(k) + 16E(k) \right) \tag{1.36}$$

and the function $\mathcal{E}_n^{S}(k,u)$ is also defined by

$$\mathcal{E}_n^{S}(k,u) := \frac{4n^2}{\pi} K(k) \cdot \left(\frac{(1-ku)^2\left((1-k^2)u+k\right)^2 K(k)}{u\left((1-2k^2)u+2k\right)\left((1-k^2)u^2+1\right)} - 4\left((1-k^2)K(k) - E(k)\right) \right). \tag{1.37}$$

(ii) $\mathcal{E}_n(k,h) = \underline{\mathcal{E}}_n(k,h)$ for $(k,h) \in \underline{\Sigma}$, where

$$\underline{\mathcal{E}}_n(k,h) := \overline{\mathcal{E}}_n(k,-h).$$

(iii) $\mathcal{E}_n(k,h) = \overline{\mathcal{E}}_n(k,h)$ for $(k,h) \in \Sigma_0$, where

$$\overline{\mathcal{E}}_n(k,h) = \frac{-16n^2}{\pi} K(k) \cdot \left((1-k^2)K(k) - E(k)\right).$$

Theorem 1.4 *Let $(\kappa(s), \mu, \nu)$ be given by Theorem 1.1 with (1.3) and $h \neq 0$ and*

$$M_n(k,h) := \frac{2\mu\pi^2 + n\pi \displaystyle\int_0^{\pi/n)} \kappa(s)^2 ds}{2\pi\omega\mu + n \displaystyle\int_0^{\pi/n} \kappa(s)^3 ds}.$$

Then $M_n(k,h)$ is given by the following (i) and (ii):
(i) $M_n(k,h) = \overline{M}_n(k,h)$ for $(k,h) \in \overline{\Sigma}$, where

$$\overline{M}_n(k,h) := \begin{cases} M_n^{S^*}(k, v(k,h)) & for\ (k,h) \in \Sigma_{S^*}, \\ M_n^{S}(k, u(k,h)) & for\ (k,h) \in \Sigma_{S}. \end{cases} \tag{1.38}$$

Here the function $M_n^{S^*}(k,v)$ is defined by

$$\underline{M_n}^{S^*}(k,v) := \frac{\sqrt{2}\pi^2}{n}\cdot\frac{\sqrt{1-v^2}\sqrt{(1-k^2)v+k^2+1}}{\big((1-k)v+1+k\big)}\cdot\frac{\varphi_1(k,v)}{\varphi_2(k,v)}\cdot\frac{1}{K(k)^2}, \quad (1.39)$$

$$\begin{aligned}\varphi_1(k,v) \ &:= \big(-(1-k^2)(1-v)^2+4\big)^2 K(k) \qquad (1.40)\\ &\quad -8(1-v^2)\big((1-k^2)v+k^2+1\big)E(k),\\ \varphi_2(k,v) \ &:= \big((1+k)v+1-k\big)\big((1-k^2)(v+1)^2+4k^2\big)\cdot \qquad (1.41)\\ &\quad \big(-(1-k^2)(1-v)^2+4\big)\end{aligned}$$

and the function $M_n^S(k,u)$ is also defined by

$$M_n^S(k,u) := \frac{-\pi^2}{2n}\cdot\frac{\sqrt{u}\sqrt{(1-k^2)u^2+1}\sqrt{(1-2k^2)u+2k}}{(1-ku)\big((1-k^2)u+k\big)}\cdot\frac{\varphi_3(k,u)}{\varphi_4(k,u)}\cdot\frac{1}{K(k)^2}, \quad (1.42)$$

$$\begin{aligned}\varphi_3(k,u) \ &:= -\Big((1-ku)^2\big((1-k^2)u+k\big)^2 \qquad (1.43)\\ &\quad +u\big((1-2k^2)u+2k\big)\big((1-k^2)u^2+1\big)\Big)K(k)\\ &\quad +2u\big((1-2k^2)u+2k\big)\big((1-k^2)u^2+1\big)E(k),\\ \varphi_4(k,u) \ &:= k^2(1-ku)^2\big((1-k^2)u^2+1\big) \qquad (1.44)\\ &\quad +u\big((1-2k^2)u+2k\big)\big((1-k^2)u+k\big)^2.\end{aligned}$$

(ii) $M_n(k,h) = \underline{M}_n(k,h)$ for $(k,h)\in\underline{\Sigma}$, where

$$\underline{M}_n(k,h) = -\overline{M}_n(k,-h).$$

For given M, the solutions of transcendental equations

$$Z(k,h) = \frac{\omega\pi}{n},\ M_n(k,h) = M \quad (1.45)$$

give the solution of (P_n^ω) by Theorem 1.1.

For example, let us determine the solution $\kappa(s)$ of (P_1^0). Figure 4 shows 1-mode solution which is obtained by solving (1.45) with $\omega = 0$ and $n = 1$. Figure 5 shows the curve which is corresponding to Figure 4. The thick-line is corresponding to $0 \le s \le \pi$.

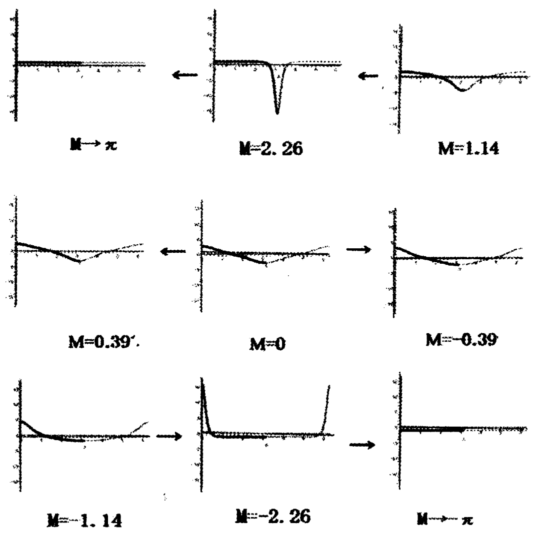

Figure 4: Curvatures of 1-mode solution for $\omega = 0$

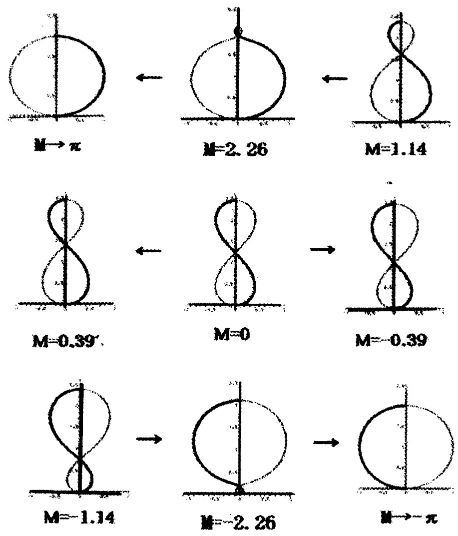

Figure 5: Curves of 1-mode solution for $\omega = 0$

We note that the other curves are not closed except for $k = k_0$ with $h = 0$ in Theorem 1.1, where k_0 is the unique solution of $2E(k) - K(k) = 0$ $(0 < k < 1)$.

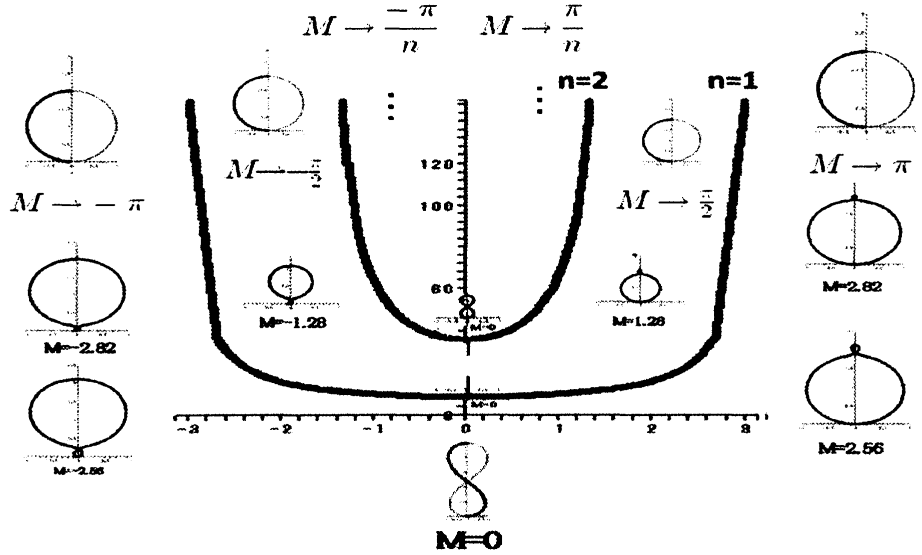

Figure 6: Energy curves of stationary solutions for $\omega = 0$

Figure 6 shows the energy curves of stationary solutions for $\omega = 0$ which are obtained from the equation (1.45) and Theorem 1.3.

Investigating the global structure, we obtain the following theorems.

Theorem 1.5 *Let $\omega = 0$ and $n \geq 1$. Then, there exists a unique n-mode solution $\kappa(s) = \kappa_n(s; M)$ of (P_n^0) for $-\frac{\pi}{n} < M < \frac{\pi}{n}$. Further there exists no solution for $M \leq -\frac{\pi}{n}$, $\frac{\pi}{n} \leq M$.*

Theorem 1.6 *Let $\omega = 0$ and $n \geq 1$. Then, there exists a unique minimizer $\kappa(s) = \kappa(s; M)$ for $-\pi < M < \pi$ with the normalizing condition $\kappa(0) := \max\limits_{0 \leq s \leq 2\pi} \kappa(s)$. This minimizer is 1-mode solution.*

Theorem 1.7 *Let $\omega = 0$ and $n \geq 1$. Then, the n-mode solution $\kappa(s) = \kappa_n(s; M)$ of (P_n^0) with property $\kappa(0) := \max_{0 \leq s \leq \pi/n} \kappa(s)$ for $0 \leq s \leq \pi/n$ satisfies the following relations:*

$$(i) \quad \lim_{M \uparrow \frac{\pi}{n}} \kappa_n(s; M) = \begin{cases} n & for\ s \in \left[0, \dfrac{\pi}{n}\right), \quad uniformaly\ in\ \left[0, \dfrac{\pi}{n}\right) \\ -\infty & for\ s = \dfrac{\pi}{n}. \end{cases}$$

$$(ii) \quad \lim_{M \downarrow -\frac{\pi}{n}} \kappa_n(s; M) = \begin{cases} \infty & s \in \left(0, \dfrac{\pi}{n}\right], \quad uniformaly\ in\ \left(0, \dfrac{\pi}{n}\right] \\ -n & s = 0. \end{cases}$$

In this paper, we show the proof of Theorem 1.1. We need long calculation to obtain Theorem 1.2 $\sim$ 1.7. The complete proofs of them will appear in a forthcoming papers.

This kind of method was first proposed by Lou-Ni-Yotsutani[5]. Later, Ikeda-Kondo-Okamoto-Yotsutani [3] and Kosugi-Morita-Yotsutani [4] developed the method.

2 Proof of Theorem 1.1

We rewrite (E_n) as first order differential equation to find the solution $\kappa(s)$.

Let us set

$$\kappa(0) := \alpha,\ \kappa\left(\frac{L}{2n}\right) := \beta\ (\alpha > 0,\ \alpha > \beta).$$

Multiplying $2\kappa_s$ to (E_n), we have

$$\frac{d}{ds}\left(\frac{d\kappa}{ds}\right)^2 = \frac{d}{ds}\left(-\frac{1}{4}\kappa^4 - \mu\kappa^2 + 2\nu\kappa\right).$$

Integrating above equation on $[0, s]$, we obtain

$$\frac{d\kappa}{ds} = -\sqrt{\tilde{g}(\kappa)}, \tag{2.1}$$

where

$$\tilde{g}(\kappa) = -\frac{1}{4}\kappa^4 - \mu\kappa^2 + 2\nu\kappa + \frac{1}{4}\alpha^4 + \mu\alpha^2 - 2\nu\alpha. \tag{2.2}$$

By the Neumann boundary condition of (E_n), we can rewrite $\tilde{g}(\kappa)$ as

$$\tilde{g}(\kappa) = \frac{1}{4}(\alpha - \kappa)(\kappa - \beta)\left(\left(\kappa + \frac{\alpha+\beta}{2}\right)^2 + 4\delta\right), \tag{2.3}$$

where δ is some constant. Comparing the coefficients of (2.2) with that of (2.3), we obtain

$$\begin{aligned}
\mu &= \frac{-1}{8}\left(3(\alpha+\beta)^2 - \frac{1}{2}(3\alpha+\beta)(\alpha+3\beta) - 8\delta\right), \\
\nu &= \frac{1}{32}(\alpha+\beta)\left((\alpha-\beta)^2 + 16\delta\right).
\end{aligned}$$

Let us set

$$A := \frac{3\alpha+\beta}{4},\ B := \frac{\alpha+3\beta}{4}.$$

Then μ and ν are represented by

$$\begin{aligned} \mu &= \frac{-1}{8}(3(A+B)^2 - 8(AB+\delta)), \\ \nu &= \frac{1}{8}(A+B)((A-B)^2+\delta). \end{aligned} \tag{2.4}$$

Further, let us set

$$\hat{\kappa} := \frac{1}{2}\left(\kappa + \frac{A+B}{2}\right). \tag{2.5}$$

Then (2.1) is represented by

$$\begin{aligned} &\frac{d\hat{\kappa}}{ds} = -\sqrt{\hat{g}(\hat{\kappa})}, \\ &\hat{\kappa}(0) = A,\ \hat{\kappa}\left(\frac{L}{2n}\right) = B, \end{aligned} \tag{2.6}$$

where

$$\hat{g}(\hat{\kappa}) = (A-\hat{\kappa})(\hat{\kappa}-B)(\hat{\kappa}^2+\delta). \tag{2.7}$$

We need to consider the following five cases in (2.6):

$$\begin{aligned} &(i) \quad A+B<0,\ \delta \le 0, \\ &(ii) \quad A+B<0,\ \delta > 0, \\ &(iii) \quad A+B>0,\ \delta > 0, \\ &(iv) \quad A+B>0,\ \delta \le 0, \\ &(v) \quad A+B=0. \end{aligned}$$

After the proof of Theorem 1.1, we obtain the following five equivalent relations:

$$\begin{aligned} &(i) \quad A+B<0,\ \delta \le 0 \Longleftrightarrow \Sigma_{S^*}, \\ &(ii) \quad A+B<0,\ \delta > 0 \Longleftrightarrow \Sigma_S, \\ &(iii) \quad A+B>0,\ \delta > 0 \Longleftrightarrow \Sigma_R, \\ &(iv) \quad A+B>0,\ \delta \le 0, \Longleftrightarrow \Sigma_{R^*}, \\ &(v) \quad A+B=0, \delta \ge 0 \Longleftrightarrow \Sigma_0, \end{aligned}$$

where Σ_{S^*}, Σ_S, Σ_{R^*}, Σ_R and Σ_0 are given by (1.8), (1.9), (1.22), (1.23) and (1.25), respectively.

We note that there exists no solution for $A+B=0$, $\delta<0$. We also note that if $(\kappa(s), \mu, \nu)$ is a solution of (E_n), then $(-\kappa(\pi/n - s), \mu, -\nu)$ is also the solution of (E_n) by (2.1) and (2.3). Hence if we have the solutions of (E_n) in the case of (i) and (ii), then we also obtain the solutions of (E_n) in the case (iv) and (iii), respectively. Thus we treat the case (i) and (ii).

2.1 Representation of solutions for $A+B<0, \delta \leq 0$

Lemma 2.1 *Suppose that $A+B<0$ and $\delta \leq 0$ in (2.6). Then the solution $\kappa(s)$ of (E_n) is represented by*

$$\kappa(s) = \kappa_n^{S^*}(s; A, B, \eta) := 2\hat{\kappa}_n^{S^*}(s; A, B, \eta) - \frac{A+B}{2} + 2\eta, \tag{2.8}$$

where $\eta := \sqrt{-\delta}$ and

$$\hat{\kappa}_n^{S^*}(s; A, B, \eta) := \frac{(A-\eta)(B-\eta)}{B - \eta + (A-B)\mathrm{cn}^2\left(\frac{n}{\pi}K(k)\left(\frac{\pi}{n} - s\right), k\right)}. \tag{2.9}$$

Moreover it holds that

$$\sqrt{(A-\eta)(B+\eta)} = \frac{2n}{\pi}K(k) \tag{2.10}$$

with

$$k = -\sqrt{\frac{2\eta(A-B)}{(A-\eta)(B+\eta)}}. \tag{2.11}$$

Proof. Under the condition that $A+B<0$ and $\delta = -\eta^2 \leq 0$ $(\eta \geq 0)$, we have

$$B < A \leq -\eta \leq 0 \leq \eta,$$

since $A > B$ and (2.7) is positive on the interval (B, A). Now we show $A \neq -\sqrt{-\delta}$. Assume that $A = -\sqrt{-\delta} < 0$. Then, substituting $\hat{\kappa} = A - 1/\xi$ into (2.6), we have

$$\begin{aligned}
\frac{L}{2n} &= \int_B^A \frac{d\hat{\kappa}}{(A-\hat{\kappa})\sqrt{-(\hat{\kappa}-B)(\hat{\kappa}+A)}} \\
&= \frac{1}{\sqrt{-2A(A-B)}} \int_{1/(A-B)}^{\infty} \frac{d\xi}{\sqrt{\left(\xi - \frac{1}{A-B}\right)\left(\xi - \frac{1}{2A}\right)}} \\
&= \frac{1}{\sqrt{-2A(A-B)}} \left[2\log\left| \sqrt{\xi - \frac{1}{A-B}} + \sqrt{\xi - \frac{1}{2A}} \right| \right]_{1/(A-B)}^{\infty} \\
&= \infty.
\end{aligned}$$

This is a contradiction. In the same way, we obtain

$$\frac{L}{2n} = \int_B^0 \frac{d\hat{\kappa}}{-\hat{\kappa}\sqrt{-\hat{\kappa}(\hat{\kappa}-B)}} = \infty$$

in the case $A = -\sqrt{\eta} = 0$. This is also contradiction. Thus it holds that

$$B < A < -\eta \leq 0 \leq \eta. \tag{2.12}$$

Let us set

$$\hat{\kappa}(s) = \frac{1}{\tilde{\kappa}(s)} + \eta. \tag{2.13}$$

Then (2.6) becomes

$$\frac{d\tilde{\kappa}}{ds} = \sqrt{(A-\eta)(B-\eta)}\sqrt{\left(\tilde{\kappa} - \frac{1}{A-\eta}\right)\left(\frac{1}{B-\eta} - \tilde{\kappa}\right)(2\eta\tilde{\kappa}+1)}.$$

Further we introduce change of variable $\tilde{\kappa}$ to φ by

$$\tilde{\kappa}(s) = \frac{1}{B-\eta} - \left(\frac{1}{B-\eta} - \frac{1}{A-\eta}\right)\sin^2\varphi(s) \quad for\ \varphi(s) \in \left[0, \frac{\pi}{2}\right]. \tag{2.14}$$

We obtain

$$\frac{d\varphi}{ds} = \frac{-1}{2}\sqrt{(A-\eta)(B+\eta)}\sqrt{1-k^2\sin^2\varphi}.$$

Integrating the above equation on $[0, s]$, we have

$$\frac{\sqrt{(A-\eta)(B+\eta)}}{2}s = K(k) - \int_0^{\varphi(s)} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \tag{2.15}$$

since $\varphi(0) = \pi/2$. At $s = \pi/n$, we obtain (2.10) by $\varphi(\pi/n) = 0$.
Substituting (2.10) and $\xi = \sin\varphi$ into (2.15), we have

$$\sin(\varphi(s)) = \mathrm{sn}\left(\frac{n}{\pi}\mathrm{K}(k)\left(\frac{\pi}{n} - s\right), k\right),$$

which implies that

$$\tilde{\kappa}(s) \;=\; \frac{1}{A-\eta} + \left(\frac{1}{B-\eta} - \frac{1}{A-\eta}\right)\mathrm{cn}^2\left(\frac{n}{\pi}K(k)\left(\frac{\pi}{n} - s\right), k\right). \tag{2.16}$$

On the other hand, it follows from (2.5) and (2.13) that we have

$$\kappa(s) = 2\,\hat{\kappa}(s) - \frac{A+B}{2} = \frac{2}{\tilde{\kappa}(s)} - \frac{A+B}{2} + 2\eta.$$

Thus, substituting (2.16) into above relation, the lemma holds. □

2.2 Representation of solutions for $A+B<0, \delta>0$

Lemma 2.2 *Suppose that $A+B<0, \delta>0$ in (2.6). Then the solution of $\kappa(s)$ of (E_n) is represented by*

$$\kappa(s) = \kappa_n^S(s; A, B, \delta) := 2\hat{\kappa}_n^S(s; A, B, \delta) - \frac{A+B}{2}, \tag{2.17}$$

where

$$\begin{aligned} &\hat{\kappa}_n^S(s; A, B, \delta) := \\ &\qquad \frac{AB_\delta + A_\delta B - (AB_\delta - A_\delta B)\mathrm{cn}\left(\frac{2n}{\pi}K(k)(\frac{\pi}{n} - s), k\right)}{A_\delta + B_\delta + (A_\delta - B_\delta)\mathrm{cn}\left(\frac{2n}{\pi}K(k)(\frac{\pi}{n} - s), k\right)}, \end{aligned} \tag{2.18}$$

$$A_\delta := \sqrt{A^2+\delta}, \quad B_\delta := \sqrt{B^2+\delta}. \tag{2.19}$$

Moreover it holds that

$$\sqrt{A_\delta B_\delta} = \frac{2n}{\pi}\mathrm{K}(k) \tag{2.20}$$

with

$$k = \sqrt{\frac{1}{2}\left(1 - \frac{AB+\delta}{A_\delta B_\delta}\right)}. \tag{2.21}$$

Proof. Let us set

$$\hat{\kappa}(s) = \frac{1}{\tilde{\kappa}(s)} + B. \tag{2.22}$$

Then (2.6) becomes

$$\frac{d\tilde{\kappa}}{ds} = B_\delta\sqrt{A-B}\sqrt{\left(\tilde{\kappa} - \frac{1}{A-B}\right)\left(\tilde{\kappa}^2 + \frac{2B}{B_\delta^2}\tilde{\kappa} + \frac{1}{B_\delta^2}\right)}.$$

Further we introduce change of variable $\tilde{\kappa}$ to φ by

$$\tilde{\kappa}(s) = \frac{1}{A-B}\left(1 + \frac{A_\delta}{B_\delta}\tan^2\frac{\varphi(s)}{2}\right), \tag{2.23}$$

where $\varphi(s) \in [0, \pi]$. We get

$$\begin{aligned} &\frac{A_\delta}{(A-B)B_\delta}\tan\frac{\varphi}{2}\left(1+\tan^2\frac{\varphi}{2}\right)\frac{d\varphi}{ds} \\ &= \sqrt{A_\delta B_\delta}\cdot\frac{A_\delta}{(A-B)B_\delta}\tan\frac{\varphi}{2}\sqrt{1+\tan^4\frac{\varphi}{2} + 2\frac{AB+\delta}{A_\delta B_\delta}\tan^2\frac{\varphi}{2}} \\ &= \sqrt{A_\delta B_\delta}\cdot\frac{A_\delta}{(A-B)B_\delta}\cdot\tan\frac{\varphi}{2}\sqrt{\left(1+\tan^2\frac{\varphi}{2}\right)^2 - 4k^2\tan^2\frac{\varphi}{2}} \\ &= \sqrt{A_\delta B_\delta}\cdot\frac{A_\delta}{(A-B)B_\delta}\cdot\tan\frac{\varphi}{2}\left(1+\tan^2\frac{\varphi}{2}\right)\sqrt{1-k^2\sin^2\varphi}\,. \end{aligned}$$

Thus we obtain
$$\frac{d\varphi}{ds} = \sqrt{A_\delta B_\delta}\sqrt{1-k^2\sin^2\varphi}.$$

Integrating the above equation on $[0, s]$, we have

$$\sqrt{A_\delta B_\delta}s = \int_0^{\varphi(s)} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \tag{2.24}$$

by $\varphi(0) = 0$. At $s = \pi/n$, we have (2.20) by $\varphi(\pi/n) = \pi$.

Substituting (2.20) and $\xi = \sin\varphi$ into (2.24), we obtain

$$\sin\varphi(s) = \mathrm{sn}\left(\frac{2n}{\pi}\mathrm{K}(k)s, k\right),$$

which implies that

$$\cos(\varphi(s)) = \mathrm{cn}\left(\frac{2n}{\pi}\mathrm{K}(k)s, k\right).$$

Thus we have

$$\tan^2\frac{\varphi(s)}{2} = \frac{1-\cos\varphi(s)}{1+\cos\varphi(s)} = \frac{1-\mathrm{cn}\left(\frac{2n}{\pi}\mathrm{K}(k)s, k\right)}{1+\mathrm{cn}\left(\frac{2n}{\pi}\mathrm{K}(k)s, k\right)}.$$

Substituting above relation into (2.23), $\tilde{\kappa}(s)$ becomes

$$\tilde{\kappa}(s) = \frac{A_\delta + B_\delta - (A_\delta - B_\delta)\,\mathrm{cn}\left(\frac{2n}{\pi}\mathrm{K}(k)s, k\right)}{(A-B)B_\delta\left(1+\mathrm{cn}\left(\frac{2n}{\pi}\mathrm{K}(k)s, k\right)\right)}. \tag{2.25}$$

On the other hand, we obtain

$$\kappa(s) = 2\,\hat{\kappa}(s) - \frac{A+B}{2} = 2\left(\frac{1}{\tilde{\kappa}(s)} + B\right) - \frac{A+B}{2}$$

by (2.5) and (2.22). Thus, substituting (2.25) into above relation, we obtain (2.17) since

$$\mathrm{cn}\left(\frac{2n}{\pi}K(k)s, k\right) = -\mathrm{cn}\left(\frac{2n}{\pi}K(k)\left(\frac{\pi}{n} - s\right), k\right).$$

□

2.3 Change of parameters for $A+B<0, \delta \leq 0$

Let us consider the case $A+B<0, \delta \leq 0$. It is difficult for us to investigate the global structure by using the parameters A, B and $\eta := \sqrt{-\delta}$. A and B belong to semi-infinite interval and η is constrained by (2.20) and (2.21). Thus we change the parameter.

Let us see $(k, \tilde{h})$ be known and A, B and η be the solutions of the system of

$$\begin{cases} k^2 = \dfrac{2\eta(A-B)}{(A-\eta)(B+\eta)}, & (2.26)\\ \sqrt{(A-\eta)(B+\eta)} = \dfrac{2n}{\pi}K(k), & (2.27)\\ A = (1-\tilde{h})B \ (0<\tilde{h}<2). & (2.28) \end{cases}$$

Then we obtain the following lemma:

Lemma 2.3 *Suppose that $A+B<0$ and $\delta \leq 0$. Then A, B and η are represented by*

$$\begin{aligned} A &= -\frac{\left((1-k^2)v+1\right)\sqrt{1-v}}{\sqrt{2}\sqrt{v+1}\sqrt{(1-k^2)v+k^2+1}} \cdot \left(\frac{2n}{\pi}K(k)\right), \\ B &= -\frac{(2-k^2)v+k^2+2}{\sqrt{2}\sqrt{1-v^2}\sqrt{(1-k^2)v+k^2+1}} \cdot \left(\frac{2n}{\pi}K(k)\right), \qquad (2.29) \\ \eta &= \frac{k^2\sqrt{1-v}}{\sqrt{2}\sqrt{1+v}\sqrt{(1-k^2)v+k^2+1}} \cdot \left(\frac{2n}{\pi}K(k)\right) \end{aligned}$$

and $v = v(k,h)$ for $(k,h) \in \Sigma_{S^*}$, where Σ_{S^*} and $v(k,h)$ are defined by (1.8) and (1.16), respectively.

Proof. It follows from (2.26),(2.27) and (2.28) that we obtain

$$\eta = \frac{-k^2}{2\tilde{h}B}\left(\frac{2n}{\pi}K(k)\right)^2. \qquad (2.30)$$

Substituting (2.28) and (2.30) into (2.27), we have

$$(1-\tilde{h})B^4 - \frac{1}{2}(2-k^2)\left(\frac{2n}{\pi}K(k)\right)^2 B^2 - \frac{k^4}{4\tilde{h}^2}\left(\frac{2n}{\pi}K(k)\right)^4 = 0.$$

If $\tilde{h} \geq 1$, then the left hand side of above equation is negative. Thus, we may consider $0<\tilde{h}<1$. Solving the above equation with respect to B, we obtain

$$\begin{aligned} &A = -(1-\tilde{h})\xi\left(\frac{2n}{\pi}K(k)\right), B = -\xi\left(\frac{2n}{\pi}K(k)\right), \\ &\eta = \frac{k^2}{2\tilde{h}\xi}\left(\frac{2n}{\pi}K(k)\right) \end{aligned} \qquad (2.31)$$

since $B < 0$, where

$$\xi := \frac{\sqrt{(2-k^2)\tilde{h} + \sqrt{(2-k^2)^2\tilde{h}^2 + 4k^4(1-\tilde{h})}}}{2\sqrt{\tilde{h}}\sqrt{1-\tilde{h}}}$$
$$for\ (k,\tilde{h}) \in \{(k,\tilde{h}); 0 < k < 1,\ 0 < \tilde{h} < 1\}.$$

To simplify the representation, we set

$$v = \frac{-2 + (2-k^2)\tilde{h} + \sqrt{(2-k^2)^2\tilde{h}^2 + 4k^4(1-\tilde{h})}}{2(1-k^2)},$$

which implies that

$$2(1-k^2)v + 2 - (2-k^2)\tilde{h} = \sqrt{(2-k^2)^2\tilde{h}^2 + 4k^4(1-\tilde{h})}. \tag{2.32}$$

Solving (2.32) with respect to $\tilde{h}$ yields

$$\tilde{h} = \frac{(v+1)\big((1-k^2)v + k^2 + 1\big)}{(2-k^2)v + k^2 + 2}.$$

Hence we have

$$(2-k^2)\tilde{h} + \sqrt{(2-k^2)^2\tilde{h}^2 + 4k^4(1-\tilde{h})} = 2(1-k^2)v + 2,$$
$$1 - \tilde{h} = \frac{(1-v)((1-k^2)v + 1)}{(2-k^2)v + k^2 + 2}$$

by (2.32). Thus ξ becomes

$$\xi = \frac{(2-k^2)v + k^2 + 2}{\sqrt{2}\sqrt{1-v^2}\sqrt{(1-k^2)v + k^2 + 1}}.$$

Substituting $h = \tilde{h} + 2$ and above relation into (2.31), the lemma holds. □

2.4 Change of parameters for $A + B < 0, \delta > 0$

Let us consider the case $A + B < 0, \delta > 0$. It is also difficult for us to investigate the global structure by using the parameters A, B and δ. Thus we change the parameters.

Let $(k, \tilde{h})$ be known and A, B and δ be the solutions of the system of

$$\begin{cases} k^2 = \dfrac{1}{2}\left(1 - \dfrac{AB+\delta}{\sqrt{(A^2+\delta)(B^2+\delta)}}\right), & (2.33) \\ \sqrt[4]{(A^2+\delta)(B^2+\delta)} = \dfrac{2n}{\pi}K(k), & (2.34) \\ A = (1-\tilde{h})B \ (0 < \tilde{h} < 2). & (2.35) \end{cases}$$

Then we obtain the following lemma:

Lemma 2.4 *Suppose that $A + B < 0$, $\delta > 0$. Then A, B and δ are represented by*

$$\begin{aligned} A &= -\frac{\sqrt{u}(1-2k^2-2k(1-k^2)u)}{\sqrt{(1-k^2)u^2+1}\sqrt{(1-2k^2)u+2k}} \cdot \left(\frac{2n}{\pi}K(k)\right), \\ B &= -\frac{\sqrt{(1-2k^2)u+2k}}{\sqrt{u}\sqrt{(1-k^2)u^2+1}} \cdot \left(\frac{2n}{\pi}K(k)\right), \qquad (2.36) \\ \delta &= \frac{(1-k^2)u((1-2k^2)u+2k)}{(1-k^2)u^2+1} \cdot \left(\frac{2n}{\pi}K(k)\right)^2 \end{aligned}$$

and $u = u(k,h)$ for $(k,h) \in \Sigma_S$, where $u(k,h)$ and Σ_S are defined by (1.9) and (1.20), respectively.

Proof. It follows from (2.34) and (2.33) that we obtain

$$\delta = (1-2k^2)\left(\frac{2n}{\pi}K(k)\right)^2 - (1-\tilde{h})B^2. \qquad (2.37)$$

Substituting (2.35) and (2.37) into (2.34), we have

$$-\tilde{h}^2(1-\tilde{h})B^4 + (1-2k^2)\tilde{h}^2\left(\frac{2n}{\pi}K(k)\right)^2 B^2 - 4k^2(1-k^2)\left(\frac{2n}{\pi}K(k)\right)^4 = 0.$$

Solving above equation with respect to B, we obtain the following two solutions (i) and (ii):

$$(i)\ B = \frac{-2\sqrt{2}k\sqrt{1-k^2}}{\sqrt{\tilde{h}}\sqrt{(1-2k^2)\tilde{h} + \sqrt{\tilde{h}^2 - 4k^2(1-k^2)(2-\tilde{h})^2}}} \cdot \left(\frac{2n}{\pi}K(k)\right)$$

$$\text{for } (k,\tilde{h}) \in \{(k,\tilde{h});\ 0 < k \le 1/\sqrt{2},\ H(k) < \tilde{h} < 2\}$$
$$\cup\{(k,\tilde{h});\ 1/\sqrt{2} < k \le 1,\ 1 < \tilde{h} < 2\},$$

$$(ii)\ B = \frac{-2\sqrt{2}k\sqrt{1-k^2}}{\sqrt{\tilde{h}}\sqrt{(1-2k^2)\tilde{h} - \sqrt{\tilde{h}^2 - 4k^2(1-k^2)(2-\tilde{h})^2}}} \cdot \left(\frac{2n}{\pi}K(k)\right)$$

$$\text{for } (k,\tilde{h}) \in \{(k,\tilde{h});\ 0 < k \le 1/\sqrt{2},\ H(k) < \tilde{h} < 1\},$$

since $B < 0$, where

$$H(k) := \frac{4k\sqrt{1-k^2}}{1+2k\sqrt{1-k^2}}.$$

Further changing the parameter $(k, \tilde{h})$ to (k, h) by

$$h = \begin{cases} 2 - \sqrt{\tilde{h}^2 - 4k^2(1-k^2)(2-\tilde{h})^2} & \textit{for case (i)}, \\ 2 + \sqrt{\tilde{h}^2 - 4k^2(1-k^2)(2-\tilde{h})^2} & \textit{for case (ii)}, \end{cases}$$

$\tilde{h}$ becomes

$$\tilde{h} = \frac{(2-h)^2 + 16k^2(1-k^2)}{8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}} \qquad (2.38)$$
$$\textit{for} (k, h) \in \Sigma_S,$$

where Σ_S is defined by (1.9).

To simplify the representation, we set

$$\begin{aligned} u &= \frac{1}{4k(1-k^2)} \cdot \Bigg(2 - h + \\ &\qquad \frac{(1-2k^2)\big((2-h)^2 + 16k^2(1-k^2)\big)}{8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}}\Bigg) \\ &= \frac{1}{4k(1-k^2)} \cdot \Bigg(2 - h + \\ &\qquad \frac{-8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}}{1-2k^2}\Bigg), \end{aligned} \qquad (2.39)$$

which implies that

$$(1-2k^2)(4k(1-k^2)u-2+h)+8k^2(1-k^2) = \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}.$$

Solving the above equation with respect to h yields

$$h = \frac{2(1-ku)((1-k^2)(1-2k^2)u + k(3-2k^2))}{(1-2k^2)u + 2k}.$$

Substituting the above relation into (2.38) gives

$$\begin{aligned} \tilde{h} &= \frac{4k(1-k^2)u - (2-h)}{(1-2k^2)} \\ &= \frac{2k((1-k^2)u^2 + 1)}{(1-2k^2)u + 2k} \end{aligned}$$

by (2.39). Hence we have

$$1-\tilde{h}=\frac{u(1-2k^2-2k(1-k^2)u)}{(1-2k^2)u+2k}.$$

Further we obtain

$$(1-2k^2)\tilde{h}+\sqrt{\tilde{h}^2-4k^2(1-k^2)(2-\tilde{h})^2}=4k(1-k^2)u$$

by (2.38) and (2.39) in the case (i). We also obtain

$$(1-2k^2)\tilde{h}-\sqrt{\tilde{h}^2-4k^2(1-k^2)(2-\tilde{h})^2}=4k(1-k^2)u.$$

in the case (ii). Using above relations, the lemma holds. $\square$

Proof of Theorem 1.1. Substituting (2.29) and (2.36) into (2.4), (2.8) and (2.17), we obtain (i) of Theorem 1.1.

We obtain (ii) of Theorem 1.1 since if $(\kappa(s), \mu, \nu)$ is a solution of (E_n), then $(-\kappa(\pi/n - s), \mu, -\nu)$ is also the solution of (E_n) by (2.1) and (2.3).

It follows from Lemma 2.4 that $A + B = 0, \delta \geq 0$ is equivalent to Σ_0. Thus we obtain (iii) of Theorem 1.1 since $u(k, 0) = 1/k$. $\square$

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